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EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 AND ADDITIONAL INTRINSIC CHARACTERISTICS IN A HOMOGENEOUS MAGNETIC FIELD

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With the use of the general covariant matrix 10-dimensional Petiau – Duffin – Kemmer formalism in cylindrical coordinates exact solutions of the quantum-mechanical equation for a particle with spin 1 in the presence of an external homogeneous magnetic field are constructed. Three linearly independent types of solutions are separated; in each case the formula for the energy levels has been found. Within similar technique for the quantum-mechanical equation for a particle with spin 1 and additional intrinsic electromagnetic characteristics – polarizability, exact solutions are found in the presence of an external homogeneous magnetic field.

Key words: spin 1, tetrad formalism, magnetic field, quantum mechanics, exact solutions, intrinsic electromagnetic structure, differential equations

1 Introduction

The problem of a quantum-mechanical particle in an external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger's) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli's and relativistic Dirac's) with spin 1/2 (the first investigation were [1, 2, 3, 4]). In the case of spin 1 particle, the most popular quantum-mechanical problem is the Coulomb one [4].

In the first part of the paper (Sections **1 – 3**), exact solutions for a ordinary vector particle will be constructed. In the second part (Sections **4 – 6**), the exact solutions for a particle with spin 1 and an additional intrinsic electromagnetic parameter (polarizability) will be constructed explicitly as well. In principle, these results provide us with a possibility for experimental testing of this characteristics – polarizability of the spin 1 particle.

To treat the problem for an ordinary vector particle we take the matrix Petiau – Duffin – Kemmer approach extended to a general covariant form on the basis of the tetrad formalism (recent consideration and references see e.g., in [5, 6]). The main equation in the tetrad form reads [6]

$$\left[i \beta^\alpha(x) (\partial_\alpha + B_\alpha - i \frac{e}{\hbar} A_\alpha) - \frac{Mc}{\hbar} \right] \Psi(x) = 0 ,$$

$$\beta^\alpha(x) = \beta^a e_{(a)}^\alpha(x) , \quad B_\alpha(x) = \frac{1}{2} J^{ab} e_{(a)}^\beta \nabla_\alpha e_{(b)\beta} ; \quad (1)$$

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$e_{(a)}^\alpha(x)$ is a tetrad, J^{ab} stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note Mc/\hbar as M). The homogeneous magnetic field $\mathbf{B} = (0, 0, B)$ corresponds to 4-potential $A^a = (0, \frac{1}{2} \vec{B} \times \vec{r})$; in the cylindric coordinates, and the last is given by

$$dS^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad A_\phi = -Br^2/2. \quad (2)$$

Choosing a diagonal cylindric tetrad

$$e_{(0)}^\alpha = (1, 0, 0, 0), \quad e_{(1)}^\alpha = (0, 1, 0, 0), \quad e_{(2)}^\alpha = (0, 0, \frac{1}{r}, 0), \quad e_{(3)}^\alpha = (0, 0, 0, 1),$$

after simple calculations, the main equation (1) reduces to the form

$$\left[i\beta^0 \partial_0 + i\beta^1 \partial_r + i\frac{\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M \right] \Psi = 0. \quad (3)$$

For brevity we will note $(eB/2\hbar)$ as B . It is better to choose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. These matrices are given in [6].

2 Separation of variables

With the use of a special substitution (it corresponds to diagonalization of the third projections of momentum P_3 and angular momentum J_3 for a particle with spin 1, specified to the cylindric tetrad basis)

$$\Psi = e^{-i\epsilon t} e^{im\phi} e^{ikz} \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix}, \quad (4)$$

the main equation reads

$$\left[\epsilon\beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \right] \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0;$$

after calculations we arrive at the radial system of 10 equations

$$\begin{aligned} -\hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 &= M\Phi_0, \\ -i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 + i\epsilon E_2 &= M\Phi_2, \\ i\hat{a}_m H_2 + i\epsilon E_1 - k H_1 &= M\Phi_1, \\ -i\hat{b}_m H_2 + i\epsilon E_3 + k H_3 &= M\Phi_3, \end{aligned} \quad (5)$$

$$\begin{aligned} \hat{a}_m \Phi_0 - i\epsilon \Phi_1 &= ME_1, & -i\hat{a}_m \Phi_2 + k \Phi_1 &= MH_1, \\ \hat{b}_m \Phi_0 - i\epsilon \Phi_3 &= ME_3, & i\hat{b}_m \Phi_2 - k \Phi_3 &= MH_3, \\ -i\epsilon \Phi_2 - ik \Phi_0 &= ME_2, & i\hat{b}_{m-1} \Phi_1 - i\hat{a}_{m+1} \Phi_3 &= MH_2, \end{aligned} \quad (6)$$

where special abbreviations were used for first order differential operators

$$\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m + Br^2}{r}\right) = \hat{a}_m, \quad \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m + Br^2}{r}\right) = \hat{b}_m.$$

From (5) – (6) it follows 4 equations for the components Φ_a

$$\begin{aligned} & (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 \\ & \quad + i\epsilon (\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) = 0, \\ & (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 \\ & \quad - ik (\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) = 0, \\ & (-\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + \hat{a}_m \hat{a}_{m+1} \Phi_3 \\ & \quad + i\epsilon \hat{a}_m \Phi_0 + ik \hat{a}_m \Phi_2 = 0, \\ & (-\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + \hat{b}_m \hat{b}_{m-1} \Phi_1 + \\ & \quad + i\epsilon \hat{b}_m \Phi_0 + ik \hat{b}_m \Phi_2 = 0; \end{aligned} \tag{7}$$

3 General analysis of the radial equations

Eqs. (7) can be transformed to the form

$$\begin{aligned} & [-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2] (k\Phi_0 + \epsilon\Phi_2) = 0, \\ & [-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2] (\epsilon\Phi_0 + k\Phi_2) \\ & \quad = (\epsilon^2 - k^2) [(\epsilon\Phi_0 + k\Phi_2) - (i\hat{b}_{m-1} \Phi_1 + i\hat{a}_{m+1} \Phi_3)]; \\ & (-\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + \hat{a}_m \hat{a}_{m+1} \Phi_3 + i\epsilon \hat{a}_m \Phi_0 + ik \hat{a}_m \Phi_2 = 0, \\ & (-\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + \hat{b}_m \hat{b}_{m-1} \Phi_1 + i\epsilon \hat{b}_m \Phi_0 + ik \hat{b}_m \Phi_2 = 0. \end{aligned} \tag{8}$$

Let us introduce new variables

$$F(r) = k \Phi_0(r) + \epsilon \Phi_2(r), \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r), \tag{10}$$

then Eqs. (8) – (9) read

$$\begin{aligned} & [-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - M^2 - k^2] F = 0, \\ & [-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - M^2] G \\ & \quad = -(\epsilon^2 - k^2) (i\hat{b}_{m-1} \Phi_1 + i\hat{a}_{m+1} \Phi_3), \end{aligned} \tag{11}$$

$$\begin{aligned} & (-\hat{a}_m \hat{b}_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + \hat{a}_m \hat{a}_{m+1} \Phi_3 + i\hat{a}_m G = 0, \\ & (-\hat{b}_m \hat{a}_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + \hat{b}_m \hat{b}_{m-1} \Phi_1 + i\hat{b}_m G = 0. \end{aligned} \tag{12}$$

For equations (12), let us multiply the first one (from the left) by \hat{b}_{m-1} and the second one by the \hat{a}_{m+1} , that results in

$$\begin{aligned} & -\hat{b}_{m-1} \hat{a}_m (\hat{b}_{m-1} \Phi_1) + (\epsilon^2 - k^2 - M^2) (\hat{b}_{m-1} \Phi_1) \\ & \quad + \hat{b}_{m-1} \hat{a}_m (\hat{a}_{m+1} \Phi_3) + i\hat{b}_{m-1} \hat{a}_m G = 0, \\ & -\hat{a}_{m+1} \hat{b}_m (\hat{a}_{m+1} \Phi_3) + (\epsilon^2 - M^2 - k^2) (\hat{a}_{m+1} \Phi_3) \\ & \quad + \hat{a}_{m+1} \hat{b}_m (\hat{b}_{m-1} \Phi_1) + i\hat{a}_{m+1} \hat{b}_m G = 0. \end{aligned} \tag{13}$$

Again, let us introduce two new field variables

$$\hat{b}_{m-1}\Phi_1 = Z_1 , \quad \hat{a}_{m+1}\Phi_3 = Z_3 ; \quad (14)$$

Eqs. (13) read as follows

$$\begin{aligned} -\hat{b}_{m-1}\hat{a}_m Z_1 + (\epsilon^2 - k^2 - M^2)Z_1 + \hat{b}_{m-1}\hat{a}_m Z_3 + i\hat{b}_{m-1}\hat{a}_m G &= 0 , \\ -\hat{a}_{m+1}\hat{b}_m Z_3 + (\epsilon^2 - M^2 - k^2)Z_3 + \hat{a}_{m+1}\hat{b}_m Z_1 + i\hat{a}_{m+1}\hat{b}_m G &= 0 . \end{aligned} \quad (15)$$

With the help of new functions $f(r), g(r)$

$$Z_1 = \frac{f+g}{2} , \quad Z_3 = \frac{f-g}{2} , \quad Z_1 + Z_3 = f , \quad Z_1 - Z_3 = g ; \quad (16)$$

the system (15) is transformed to the following form

$$\begin{aligned} -\hat{b}_{m-1}\hat{a}_m g + (\epsilon^2 - k^2 - M^2)\frac{f+g}{2} + i\hat{b}_{m-1}\hat{a}_m G &= 0 , \\ \hat{a}_{m+1}\hat{b}_m g + (\epsilon^2 - M^2 - k^2)\frac{f-g}{2} + i\hat{a}_{m+1}\hat{b}_m G &= 0 . \end{aligned} \quad (17)$$

Combining these equations we get

$$\begin{aligned} [-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2]g + i(\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m)G &= 0 , \\ (-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m)g + (\epsilon^2 - k^2 - M^2)f + i(\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m)G &= 0 . \end{aligned} \quad (18)$$

In turn, Eqs. (11) can be presented as

$$\begin{aligned} (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - M^2 - k^2) F &= 0 , \\ (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2) G &= -i(\epsilon^2 - k^2) f . \end{aligned} \quad (19)$$

Further, with the use of identities

$$-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m = \Delta , \quad -\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m = 2B \quad (20)$$

Eqs. (19) and (18) can be written as follows

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0 , \\ \Delta G &= M^2 G - i(\epsilon^2 - k^2) f , \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G , \\ (\epsilon^2 - k^2 - M^2) f - i\Delta G + 2B g &= 0 . \end{aligned} \quad (21)$$

With the help of the second equation, from the forth one it follows

$$f = -i G + \frac{2B}{M^2} g . \quad (22)$$

Now, one excludes the function f in the second equation in (21) and gets

$$(\Delta + \epsilon^2 - k^2 - M^2) G = -i(\epsilon^2 - k^2) \frac{2B}{M^2} g . \quad (23)$$

Thus, the general problem is reduced to the system of four equations

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) F &= 0 , \\ f &= -i G + \frac{2B}{M^2} g , \\ (\Delta + \epsilon^2 - k^2 - M^2) g &= 2iB G , \\ (\Delta + \epsilon^2 - k^2 - M^2) G &= -2iB \frac{\epsilon^2 - k^2}{M^2} g . \end{aligned} \quad (24)$$

The structure of this system allows to separate an evident linearly independent solution as follows

$$\begin{aligned} f(r) &= 0 , & g(r) &= 0 , & H(r) &= 0 , \\ F(r) &\neq 0 , & (\Delta - k^2 - M^2 + \epsilon^2) F &= 0 . \end{aligned} \quad (25)$$

Corresponding functions and energy spectrum are known. We are to solve the system of two last equations in (24); in matrix form it reads (let $\gamma = (\epsilon^2 - k^2)/M^2$)

$$(\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g \\ G \end{vmatrix} = \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} \begin{vmatrix} g \\ G \end{vmatrix} . \quad (26)$$

Let us construct the transformation changing the matrix on the right to a diagonal form

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} &= \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix} , \\ \begin{vmatrix} g' \\ G' \end{vmatrix} &= S \begin{vmatrix} g \\ G \end{vmatrix} , & S &= \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} . \end{aligned} \quad (27)$$

The problem reduces to linear systems

$$\begin{cases} -\lambda_1 s_{11} - 2iB\gamma s_{12} = 0 , \\ 2iB s_{11} - \lambda_1 s_{12} = 0 , \end{cases} \quad \begin{cases} -\lambda_2 s_{21} - 2iB\gamma s_{22} = 0 , \\ 2iB s_{21} - \lambda_2 s_{22} = 0 . \end{cases}$$

The values of λ_1 and λ_2 are given by

$$\begin{aligned} \lambda_1 &= +2B\sqrt{\gamma} , & \lambda_2 &= -2B\sqrt{\gamma} , \\ i s_{11} - \sqrt{\gamma} s_{12} &= 0 , & i s_{21} + \sqrt{\gamma} s_{22} &= 0 , \\ s_{12} = 1 , s_{22} = 1 , & S &= \begin{vmatrix} -i\sqrt{\gamma} & 1 \\ +i\sqrt{\gamma} & 1 \end{vmatrix} . \end{aligned} \quad (28)$$

In the new (primed) basis, Eqs. (26) take the form of two separated differential equations

$$\begin{aligned} (\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma}) g' &= 0 , \\ (\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma}) G' &= 0 ; \end{aligned} \quad (29)$$

Recalling the meaning of Δ , let us specify the second order differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 ,$$

$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma}, \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} . \quad (30)$$

It is convenient to introduce a new variable $x = Br^2$, then Eq. (30) reads³

$$x \frac{d^2 \varphi}{dx^2} + \frac{d\varphi}{dx} - \left(\frac{m^2}{4x} + \frac{x}{4} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) \varphi = 0 . \quad (31)$$

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[\frac{A^2 - m^2/4}{x} + (C^2 - \frac{1}{4})x - 2AC - C - \frac{m}{2} + \frac{\lambda^2}{4B} \right] f = 0 .$$

When A, C are taken as $A = + |m|/2$, $C = +1/2$ the previous equation becomes simpler

$$x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left(A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) R = 0 ,$$

which is of confluent hypergeometric type

$$x Y'' + (\gamma - x) Y' - \alpha Y = 0 ,$$

$$\alpha = \frac{|m|}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}, \quad \gamma = |m| + 1 .$$

To obtain polynomials we must impose an additional condition $\alpha = -n$; which provides us with the following quantization rule for λ^2

$$\lambda^2 = 4B \left(n + \frac{1}{2} + \frac{|m| + m}{2} \right) . \quad (32)$$

Thus, we have arrived at two formulas for the energy

$$\sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} ,$$

$$\sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M} . \quad (33)$$

In turn, the energy spectrum for the case (76) is given by

$$\epsilon^2 = M^2 + k^2 + \lambda^2 \quad (34)$$

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and the corresponding energy spectra are found.

³For definiteness let us consider B to be positive, which does not affect the generality of the analysis. So, to infinite values of r correspond infinite and positive values of x .

4 On a spin 1 particle with intrinsic structure – polarizability

In [7]–[13], it was described a generalized equation for spin 1 particle possessing in addition to electric charge the special electromagnetic characteristics named polarizability. In the frame of the first order relativistic wave equations, such a particle requires a 15-dimensional wave function, consisting of a 4-vector $\Phi_a(x)$, 4-tensor $\Phi_{ab}(x)$, and subsidiary scalar and 4-vector fields, $C(x)$ and $C_a(x)$.

To treat the problem we take the matrix approach in the theory of the generalized $S = 1$ particle extended to a general covariant form on the base of tetrad formalism (recent consideration, notation and list of references see in [15, 16]). The use of cylindric tetrad permits to take account of the cylindric symmetry of the problem. The main equation in tetrad form is [12]

$$\left[\Gamma^0 \partial_0 + \Gamma^1 \partial_r + \frac{1}{r} \Gamma^2 (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + \Gamma^3 \partial_z - M \right] \Psi = 0. \quad (35)$$

It is better to choose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. These matrices Γ^a are given in [6].

5 Separation of variables

With the use of special substitution

$$\begin{aligned} \Psi = \{ C, C_0, \vec{C}, \Phi_0, \vec{\Phi}, \vec{E}, \vec{H} \}, \quad C(x) &= e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} C(r), \\ C_0 &= e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} C_0(r), \quad \vec{C} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{vmatrix} C_1(r) \\ C_2(r) \\ C_3(r) \end{vmatrix}, \\ \Phi_0 &= e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \Phi_0(r), \quad \vec{\Phi} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{vmatrix} \Phi_1(r) \\ \Phi_2(r) \\ \Phi_3(r) \end{vmatrix}, \\ \vec{E} &= e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{vmatrix} E_1(r) \\ E_2(r) \\ E_3(r) \end{vmatrix}, \quad \vec{H} = e^{-i\epsilon t/\hbar} e^{ikz} e^{im\phi} \begin{vmatrix} H_1(r) \\ H_2(r) \\ H_3(r) \end{vmatrix}, \end{aligned} \quad (36)$$

after calculations we arrive at the radial system of 15 equations

$$-i\epsilon C_0 - \hat{b}_{m-1} C_1 - \hat{a}_{m+1} C_3 - ik C_2 = M C, \quad (37)$$

$$-\hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 = M C_0,$$

$$i\epsilon E_1 + i\hat{a}_m H_2 - ik H_1 = M C_1,$$

$$i\epsilon E_2 - i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 = M C_2,$$

$$i\epsilon E_3 - i\hat{b}_m H_2 + k H_3 = M C_3, \quad (38)$$

$$-i\epsilon \sigma C - \hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 = M \Phi_0,$$

$$i\epsilon E_1 - \sigma \hat{a}_m C + i\hat{a}_m H_2 - k H_1 = M \Phi_1,$$

$$i\epsilon E_2 - i\hat{b}_{m-1} H_1 + i\hat{a}_{m+1} H_3 + i k \sigma C = M \Phi_2,$$

$$i\epsilon E_3 - \sigma \hat{b}_m C - i\hat{b}_m H_2 + k H_3 = M \Phi_3, \quad (39)$$

$$\begin{aligned}
-i\epsilon \Phi_1 + \hat{a}_m \Phi_0 &= M E_1, & -i\epsilon \Phi_2 - ik \Phi_0 &= M E_2, \\
-i\epsilon \Phi_3 + \hat{b}_m \Phi_0 &= M E_3, & -i\hat{a}_m \Phi_2 + k \Phi_1 &= M H_1, \\
i\hat{b}_{m-1} \Phi_1 - i\hat{a}_{m+1} \Phi_3 &= M H_2, & i\hat{b}_m \Phi_2 - k \Phi_3 &= M H_3.
\end{aligned} \tag{40}$$

6 Solution of the radial system

With the use of (38), Eqs. (39) give

$$\begin{aligned}
C_0 &= \Phi_0 + i \frac{\epsilon \sigma}{M} C, & C_1 &= \Phi_1 + \frac{\sigma}{M} \hat{a}_m C, \\
C_2 &= \Phi_2 - i \frac{k \sigma}{M} C, & C_3 &= \Phi_3 + \frac{\sigma}{M} \hat{b}_m C.
\end{aligned} \tag{41}$$

Substituting these formulas for C_a into (37)

$$\begin{aligned}
& -i\epsilon(\Phi_0 + i \frac{\epsilon \sigma}{M} C) - \hat{b}_{m-1}(\Phi_1 + \frac{\sigma}{M} \hat{a}_m C) \\
& - \hat{a}_{m+1}(\Phi_3 + \frac{\sigma}{M} \hat{b}_m C) - ik(\Phi_2 - i \frac{k \sigma}{M} C) = M C,
\end{aligned}$$

we further get

$$\begin{aligned}
M(\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) &= -iM(\epsilon \Phi_0 + k \Phi_2) \\
+ \sigma(-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m + \epsilon^2 - k^2)C &- M^2 C.
\end{aligned} \tag{42}$$

This equation will be required below.

Note that Eqs. (39) and (40) include the main field variables, 4-vector and 4-tensor, and also the scalar C obeying Eq. (42)

$$\begin{aligned}
-i\epsilon \sigma C - \hat{b}_{m-1} E_1 - \hat{a}_{m+1} E_3 - ik E_2 &= M \Phi_0, \\
i\epsilon E_1 - \sigma \hat{a}_m C + i \hat{a}_m H_2 - k H_1 &= M \Phi_1, \\
i\epsilon E_2 - i\hat{b}_{m-1} H_1 + \hat{a}_{m+1} H_3 + i k \sigma C &= M \Phi_2, \\
i\epsilon E_3 - \sigma \hat{b}_m C - i\hat{b}_m H_2 + k H_3 &= M \Phi_3,
\end{aligned} \tag{43}$$

$$\begin{aligned}
-i\epsilon \Phi_1 + \hat{a}_m \Phi_0 &= M E_1, & -i\epsilon \Phi_2 - ik \Phi_0 &= M E_2, \\
-i\epsilon \Phi_3 + \hat{b}_m \Phi_0 &= M E_3, & -i\hat{a}_m \Phi_2 + k \Phi_1 &= M H_1, \\
i\hat{b}_{m-1} \Phi_1 - i\hat{a}_{m+1} \Phi_3 &= M H_2, & i\hat{b}_m \Phi_2 - k \Phi_3 &= M H_3.
\end{aligned} \tag{44}$$

By means of (44), we are to eliminate tensor components in (43). Then we obtain two equations

$$\begin{aligned}
& -i\epsilon \sigma M C + (-\hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - k^2 - M^2) \Phi_0 \\
& + i\epsilon(\hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3 + ik \Phi_2) = 0,
\end{aligned} \tag{45}$$

$$\begin{aligned}
& ik\sigma M C + (\epsilon^2 - \hat{b}_{m-1} \hat{a}_m - \hat{a}_{m+1} \hat{b}_m - M^2) \Phi_2 \\
& - ik(i\epsilon \Phi_0 + \hat{b}_{m-1} \Phi_1 + \hat{a}_{m+1} \Phi_3) = 0.
\end{aligned} \tag{46}$$

Multiplying the first one (45) by $+ik$, and the second one (46) by $i\epsilon$, and summing the results, we get

$$(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2 + \epsilon^2) (k \Phi_0 + \epsilon \Phi_2) = 0 . \quad (47)$$

In the same manner, combining Eqs. (45)–(46) with other coefficients, we arrive at

$$\begin{aligned} & (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2) (\epsilon \Phi_0 + k\Phi_2) \\ & = -i(\epsilon^2 - k^2) (\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3) + i\sigma M (\epsilon^2 - k^2) C . \end{aligned} \quad (48)$$

Thus, two second order equations have been found

$$(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2 + \epsilon^2) (k \Phi_0 + \epsilon \Phi_2) = 0 , \quad (49)$$

$$\begin{aligned} & (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2) (\epsilon \Phi_0 + k\Phi_2) \\ & = -i(\epsilon^2 - k^2) (\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3) + i\sigma M (\epsilon^2 - k^2) C . \end{aligned} \quad (50)$$

Now, let us turn to equations in (43), containing functions $m\Phi_1$ and $m\Phi_3$:

$$\begin{aligned} & (-\hat{a}_m\hat{b}_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 \\ & + \hat{a}_m\hat{a}_{m+1} \Phi_3 + i\hat{a}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \hat{a}_m C = 0 ; \end{aligned} \quad (51)$$

and

$$\begin{aligned} & (-\hat{b}_m\hat{a}_{m+1} + \epsilon^2 - k^2 - M^2) \Phi_3 \\ & + \hat{b}_m\hat{b}_{m-1} \Phi_1 + i\hat{b}_m (\epsilon\Phi_0 + k\Phi_2) - M\sigma \hat{b}_m C = 0 , \end{aligned} \quad (52)$$

In two last equations, (51 and 52), multiplying the the first one by \hat{b}_{m-1} (from the left) and the second one by \hat{a}_{m+1} (from the left), we produce

$$\begin{aligned} & (-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2) \hat{b}_{m-1}\Phi_1 \\ & + \hat{b}_{m-1}\hat{a}_m\hat{a}_{m+1} \Phi_3 + i\hat{b}_{m-1}\hat{a}_m (\epsilon \Phi_0 + k\Phi_2) - M\sigma \hat{b}_{m-1}\hat{a}_m C = 0 , \end{aligned} \quad (53)$$

$$\begin{aligned} & (-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2) \hat{a}_{m+1}\Phi_3 \\ & + \hat{a}_{m+1}\hat{b}_m\hat{b}_{m-1} \Phi_1 + i\hat{a}_{m+1}\hat{b}_m (\epsilon\Phi_0 + k\Phi_2) - M\sigma \hat{a}_{m+1}\hat{b}_m C = 0 . \end{aligned} \quad (54)$$

It is better to introduce new field variables

$$\begin{aligned} F(r) &= k \Phi_0 + \epsilon \Phi_2 , & G(r) &= \epsilon \Phi_0 + k \Phi_2 , \\ \hat{b}_{m-1}\Phi_1 &= Z_1 , & \hat{a}_{m+1}\Phi_3 &= Z_3 ; \end{aligned} \quad (55)$$

then the system (53)–(54) reads

$$\begin{aligned} & (-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2) Z_1 \\ & + \hat{b}_{m-1}\hat{a}_m Z_3 + i\hat{b}_{m-1}\hat{a}_m G - M\sigma \hat{b}_{m-1}\hat{a}_m C = 0 , \end{aligned} \quad (56)$$

$$\begin{aligned} & (-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2) Z_3 \\ & + \hat{a}_{m+1}\hat{b}_m Z_1 + i\hat{a}_{m+1}\hat{b}_m G - M\sigma \hat{a}_{m+1}\hat{b}_m C = 0 . \end{aligned} \quad (57)$$

Again, it is convenient to define new variables $f(r)$, $g(r)$:

$$Z_1 = \frac{f+g}{2}, \quad Z_3 = \frac{f-g}{2}, \quad Z_1 + Z_3 = f, \quad Z_1 - Z_3 = g; \quad (58)$$

then Eqs. (56) – (57) give

$$\begin{aligned} & (-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2) \frac{f+g}{2} \\ & + \hat{b}_{m-1}\hat{a}_m \frac{f-g}{2} + i\hat{b}_{m-1}\hat{a}_m G - M\sigma \hat{b}_{m-1}\hat{a}_m C = 0, \end{aligned} \quad (59)$$

$$\begin{aligned} & (-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2) \frac{f-g}{2} \\ & + \hat{a}_{m+1}\hat{b}_m \frac{f+g}{2} + i\hat{a}_{m+1}\hat{b}_m G - M\sigma \hat{a}_{m+1}\hat{b}_m C = 0, \end{aligned} \quad (60)$$

After simple manipulation, from two last equations it follows that

$$\begin{aligned} & [-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2] g \\ & + (-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m) (-iG + M\sigma C) = 0, \\ & (-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m) g + (\epsilon^2 - k^2 - M^2)f \\ & + (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m) (-iG + M\sigma C) = 0. \end{aligned} \quad (61)$$

With the use of identities

$$-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m = \Delta, \quad -\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m = 2B$$

Eqs. (61) can be written as

$$\begin{aligned} & [\Delta + \epsilon^2 - k^2 - M^2] g + 2B (-iG + M\sigma C) = 0, \\ & 2B g + (\epsilon^2 - k^2 - M^2)f + \Delta (-iG + M\sigma C) = 0. \end{aligned} \quad (62)$$

In turn, Eqs. (47), (48) will read (in the new variables)

$$\begin{aligned} & (\Delta - k^2 - M^2 + \epsilon^2) F = 0, \\ & (\Delta - M^2) G = -i(\epsilon^2 - k^2) f + i\sigma M (\epsilon^2 - k^2) C. \end{aligned} \quad (63)$$

Let us collect results together

$$(\Delta - k^2 - M^2 + \epsilon^2) F = 0, \quad (64)$$

$$(\Delta - M^2) G = -i(\epsilon^2 - k^2) f + i\sigma M (\epsilon^2 - k^2) C, \quad (65)$$

$$(\Delta + \epsilon^2 - k^2 - M^2) g + 2B (-iG + M\sigma C) = 0, \quad (66)$$

$$2B g + (\epsilon^2 - k^2 - M^2)f + \Delta (-iG + M\sigma C) = 0. \quad (67)$$

It is possible to eliminate the function $C(r)$ in the above equation. To show how it can be done, let us turn to a couple of equations in (39), containing the terms $M \Phi_1, M \Phi_3$, and find

the combination

$$\begin{aligned}
& \hat{b}_{m-1}M\Phi_1 + \hat{a}_{m+1}\Phi_3 \\
&= i\epsilon \hat{b}_{m-1}E_1 - \sigma \hat{b}_{m-1}\hat{a}_m C + i \hat{b}_{m-1}\hat{a}_m H_2 - k \hat{b}_{m-1}H_1 \\
&+ i\epsilon \hat{a}_{m+1}E_3 - \sigma \hat{a}_{m+1}\hat{b}_m C - i\hat{a}_{m+1}\hat{b}_m H_2 + k \hat{a}_{m+1}H_3 \\
&= i\epsilon (\hat{b}_{m-1}E_1 + \hat{a}_{m+1}E_3) - \sigma (\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m) C \\
&+ i (\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m) H_2 - k (\hat{b}_{m-1}H_1 - \hat{a}_{m+1}H_3)
\end{aligned}$$

from whence, with the help of the first and third equations in (39) in the form

$$\begin{aligned}
(\hat{b}_{m-1} E_1 + \hat{a}_{m+1} E_3) &= -i\epsilon \sigma C - ik E_2 - M \Phi_0 , \\
(\hat{b}_{m-1} H_1 - \hat{a}_{m+1} H_3) &= \epsilon E_2 + k\sigma C + iM \Phi_2 ,
\end{aligned}$$

we obtain

$$\begin{aligned}
& \hat{b}_{m-1}M\Phi_1 + \hat{a}_{m+1}\Phi_3 \\
&= i\epsilon (-i\epsilon \sigma C - ik E_2 - M \Phi_0) - \sigma (\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m) C \\
&+ i (\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m) H_2 - k (\epsilon E_2 + k\sigma C + iM \Phi_2) .
\end{aligned}$$

From this, after evident calculation, we arrive at

$$\begin{aligned}
& \hat{b}_{m-1}M\Phi_1 + \hat{a}_{m+1}\Phi_3 = -iM (\epsilon \Phi_0 + k \Phi_2) \\
&+ \sigma (-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2) C - 2iB H_2
\end{aligned} \tag{68}$$

Comparing (42) and (68), we conclude that there exists a linear relation

$$2iB H_2(r) = M^2 C(r) . \tag{69}$$

Due to Eq. (40), it holds

$$i\hat{b}_{m-1} \Phi_1 - i\hat{a}_{m+1} \Phi_3 = M H_2 \implies ig = M H_2 ; \tag{70}$$

therefore, the function $C(r)$ is expressed through $g(r)$:

$$C(r) = -\frac{2B}{M^3} g(r) \tag{71}$$

The system (64) – (67), after excluding $C(r)$, takes the form

$$(\Delta - k^2 - M^2 + \epsilon^2) F = 0 , \tag{72}$$

$$(\Delta - M^2) G = -i(\epsilon^2 - k^2) f - i\sigma(\epsilon^2 - k^2) \frac{2B}{M^2} g , \tag{73}$$

$$\left[\Delta + \epsilon^2 - k^2 - M^2 - \sigma \left(\frac{2B}{M} \right)^2 \right] g - 2iB G = 0 , \tag{74}$$

$$2B g + (\epsilon^2 - k^2 - M^2)f + \Delta (-iG - \sigma \frac{2B}{M^2} g) = 0 . \tag{75}$$

The structure of the system allows to separate evident linearly independent solution as follows

$$\begin{aligned} f(r) &= 0, & g(r) &= 0, & H(r) &= 0, \\ F(r) &\neq 0, & (\Delta - k^2 - M^2 + \epsilon^2) F &= 0; \end{aligned} \quad (76)$$

corresponding functions and the energy spectrum are known (also see below).

We are to solve the system of three last equations in (73) – (75). With the help of (74)

$$\begin{aligned} \Delta G &= M^2 G - i(\epsilon^2 - k^2) f - i\sigma(\epsilon^2 - k^2) \frac{2B}{M^2} g, \\ \Delta g &= -[\epsilon^2 - k^2 - M^2 - \sigma(\frac{2B}{M})^2] g - 2iB G, \end{aligned}$$

Eq. (75) takes the form of the linear relation

$$M^2 f = i \left(-M^2 + \sigma \frac{4B^2}{M^2} \right) G + 2B \left(1 - \sigma - \frac{\sigma^2 4B^2}{M^2 M^2} \right) g. \quad (77)$$

Now, returning to Eqs. (73)–(74), after excluding the function f and using the notation

$$\gamma = \frac{\epsilon^2 - k^2}{M^2}, \quad \beta = \sigma \frac{4B^2}{M^2}, \quad \alpha = \gamma \rho, \quad \rho = 1 - \frac{4B^2 \sigma^2}{M^4};$$

we arrive at two equations

$$\begin{aligned} (\Delta + \epsilon^2 - k^2 - M^2) g &= \beta g(r) + 2iB G(r), \\ (\Delta + \epsilon^2 - k^2 - M^2) G &= -2iB\alpha g(r) + \beta\gamma G(r). \end{aligned} \quad (78)$$

In matrix form they read

$$(\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g(r) \\ G(r) \end{vmatrix} = \begin{vmatrix} \beta & 2iB \\ -2iB\alpha & \beta\gamma \end{vmatrix} \begin{vmatrix} g(r) \\ G(r) \end{vmatrix}. \quad (79)$$

Let us construct the transformation changing the matrix on the right to a diagonal form

$$\begin{aligned} (\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} &= \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix}, \\ \begin{vmatrix} g' \\ G' \end{vmatrix} &= S \begin{vmatrix} g \\ G \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}; \end{aligned} \quad (80)$$

the problem is reduced to a couple of linear systems

$$\begin{cases} (\beta - \lambda_1) s_{11} - 2iB\alpha s_{12} = 0, \\ 2iB s_{11} + (\beta\gamma - \lambda_1) s_{12} = 0, \end{cases} \quad \begin{cases} (\beta - \lambda_2) s_{21} - 2iB\alpha s_{22} = 0, \\ 2iB s_{21} + (\beta\gamma - \lambda_2) s_{22} = 0. \end{cases}$$

The eigenvalues λ_1, λ_2 are

$$\begin{aligned} \lambda_1 &= \frac{\beta(1 + \gamma) + \sqrt{\beta^2(1 - \gamma)^2 + 16B^2\rho\gamma}}{2}, \\ \lambda_2 &= \frac{\beta(1 + \gamma) - \sqrt{\beta^2(1 - \gamma)^2 + 16B^2\rho\gamma}}{2}. \end{aligned} \quad (81)$$

let it be

$$\begin{aligned} s_{12} = 1, \quad s_{22} = 1, \quad s_{11} = \frac{\lambda'_1 - \beta\gamma}{2iB}, \quad s_{21} = \frac{\lambda_2 - \beta\gamma}{2iB}, \\ g' = \frac{\lambda_1 - \beta\gamma}{2iB}g + G, \quad G' = \frac{\lambda_2 - \beta\gamma}{2iB}g + G; \end{aligned} \quad (82)$$

In the new (primed) basis, Eqs. (80) take the form of two separated differential equations

$$\begin{aligned} \left(\Delta + \epsilon^2 - k^2 - M^2 - \lambda_1 \right) g' &= 0, \\ \left(\Delta + \epsilon^2 - k^2 - M^2 - \lambda'_2 \right) G' &= 0; \end{aligned} \quad (83)$$

Recalling the meaning of Δ , let us specify the second order equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0. \quad (84)$$

This equation was examined above. We obtain two possibilities for the energy spectrum:

$$\begin{aligned} g' &\neq 0, \quad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda_1, \\ G' &\neq 0, \quad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda'_2. \end{aligned} \quad (85)$$

Both Eqs. (85) can be written as

$$M^2\gamma = \lambda^2 + M^2 + \frac{\beta(1 + \gamma) \pm \sqrt{\beta^2(1 - \gamma)^2 + 16B^2\rho\gamma}}{2}. \quad (86)$$

It is convenient to introduce new variable $x = \gamma - 1$, and also with the help of

$$\beta = \sigma \frac{4B^2}{M^2}, \quad \rho = 1 - \frac{4B^2\sigma^2}{M^4} = 1 - \frac{\beta^2}{4B^2}, \quad 16\rho B^2 = 16B^2 - 4\beta^2$$

to eliminate the parameter ρ :

$$(2M^2 - \beta)x - 2(\lambda^2 + \beta) = \pm \sqrt{\beta^2 x^2 + (16B^2 - 4\beta^2)(x + 1)}. \quad (87)$$

Thus, we get the second order equation

$$\begin{aligned} M^2(M^2 - \beta)x^2 - \left[(\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right] x \\ + (\lambda^2 + \beta)^2 - (4B^2 - \beta^2) = 0; \end{aligned} \quad (88)$$

its solutions read

$$\begin{aligned} \epsilon^2 - M^2 - k^2 = \frac{1}{2(M^2 - \beta)} \left\{ \left[(\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right] \right. \\ \left. \pm \left[\left[(\lambda^2 + \beta)(2M^2 - \beta) + (4B^2 - \beta^2) \right]^2 \right. \right. \\ \left. \left. - 4M^2(M^2 - \beta)(\lambda^2 + \beta)^2 - (4B^2 - \beta^2) \right] \right\}. \end{aligned} \quad (89)$$

Note, that the case (76) gives the following spectrum $\epsilon^2 = M^2 + k^2 + \lambda^2$, so for these solutions the polarizability does not manifest itself in a magnetic field.

Thus, on the base of general covariant formalism in the vector particle with polarizability, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and corresponding energy spectra are found.

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